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On the far wake and induced drag of aircraft

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A set of matched asymptotic expansions is proposed for the flow far behind an aircraft, with the primary purpose of identifying lift, thrust and drag, particularly induced drag, in a unified manner in integral statements of the momentum equation. The fluid in the far wake is inviscid and incompressible, and variations of total pressure are allowed, as are vortex sheets. A notable feature is that the Trefftz-plane approximation is not invoked; instead the wake is taken as fully rolled-up, and the analysis proceeds without the assumption of light loading. Attention is paid to the absolute convergence of integrals over infinite domains and handling of discontinuities. The expansion includes a sink term, which appears new, so that the mass flux through a transverse plane is non-zero, as is the flux of mechanical energy. The lift can be formally attributed to the velocity induced by the *bound* vortex of the wing, which is at odds with some treatments, although consistent with Prandtl's analysis over a ground plane. The drag contains the integral of $\rho(v^2+w^2-u^2)/2$, as in many treatments of the subject, u being the perturbation velocity along the wake. The negative sign for u^2 appears paradoxical on two counts, one of which is resolved here. First, its very presence instead of the + sign, which would lead to the perturbation kinetic energy and therefore a compelling explanation of induced drag, is explained by the longitudinal energy flux. This energy, the integral of ρu^2, is continuously provided by the unsteady starting-vortex system and was deposited earlier by the aircraft. Second, it appears that negative drag could be predicted by this equation. This is shown to be impossible, because of inequalities between the integrals of (v^2+w^2) and of u^2, but the proof is valid only if the vorticity is of only one sign on each side. A general proof of positivity has not been derived, because of nonlinearities, but neither has a counter-example.

1. Introduction

This work was motivated by long-standing dissatisfaction with the theory of induced drag, in quite a few respects. First, much of the analysis requires the wake to be undeformed; essentially, the vortex sheet has the cross-section given by the wing (and tail) trailing edges. This will be referred to as the Trefftz-plane approximation, or TFA. This has been justified with the assumption of 'light loading', best summarized by the ratio of lift coefficient to aspect ratio, C_l/AR. The roll-up of the wake is slow if $C_l/\mathit{AR} \ll 1$. This statement is correct, but not uniformly along the span. It has been shown, within the two-dimensional time-dependent model of the vortex sheet, that the rate of roll-up is infinite at the tip (Spreiter & Sacks 1951; Moore & Saffman 1973; Spalart 1998). Therefore, the TFA is not realistic, even when considering the wing's vortex sheet abeam the tail; in fact, roll-up starts before the wing's trailing edge. This rapidity of the roll-up is consistent with the numerical difficulties encountered near the tips by free-wake computational models.

Second, the classical formula for induced drag contains the integral of $\rho(v^2 + w^2 - u^2)/2$ where ρ is the density, instead of the kinetic energy $\rho(v^2 + w^2 + u^2)/2$ as might have been expected on physical grounds, and even recent treatments do not comment (Kroo 2001); the theory has not evolved since Prandtl (Prandtl & Tietjens 1934; Milne-Thomson 1958; Jones 1990). Typically, the 'transverse kinetic energy' $\rho(v^2 + w^2)/2$ is invoked, and this has no rational basis. Most of the uses of the theory have approximated the slender roll-up as a two-dimensional time evolution, which conserves the transverse kinetic energy, making the results insensitive to the TFA, but leaving the interpretation that the aircraft is incurring drag because it is depositing kinetic energy flawed. Superficially, this negative sign of u^2 allows the induced drag to be negative, suggesting that the formula is suspect. With light loading, the negative term is of higher order than the two positive ones, yet a general finding that negative drag cannot be predicted is more than desirable (although discovering negative induced drag would have been preferable for the author's career in the industry). One possibility was that the puzzling sign was a consequence of the light-loading approximation, and somehow would disappear in a general theory. It also appeared possible that the $-u^2/2$ term was not the only one at this order in a systematic expansion, because the derivation was somehow deficient, and therefore was of no value. Examples of flawed 'higher-order theories' are not rare in fluid mechanics.

These first two concerns appears to be unique to the author. Dissatisfaction with the theory in the community has centred on the disconnection between the physical reasoning for lift and that for drag, and on the failure to agree on simple statements using the principles of conservation of momentum or impulse directly (Sears 1974). In addition, there is outright disagreement over the value of the simplest and most essential of integrals: that of vertical momentum in the wake: $\rho \iint w \mathrm{d}y\, \mathrm{d}z$. This disagreement was explained by L. Wigton in internal Boeing memos of 1987 using Fubini's theorem; this improper integral (over an infinite two-dimensional domain) is not absolutely convergent, because the integrand is only of order $1/r^2$ for large r and $\iint |w| \mathrm{d}y\, \mathrm{d}z = \infty$, so that integrating along one direction and then the other can give different answers. For a pair of point vortices with circulation $\pm\Gamma$ placed at $y = \pm b_0/2$, $\int(\int w \mathrm{d}y)\, \mathrm{d}z = 0$, but $\int(\int w \mathrm{d}z)\, \mathrm{d}y = \Gamma b_0$. The latter order gives the expected amount of momentum, which has given it an artificial credence, but this integral must be considered as undeterminate. This research field reveals many instances of a derivation giving a correct answer for the wrong reasons. The inclusion of a ground plane does much to resolve these issues, by ensuring an $O(1/r^3)$ dependence and therefore absolute convergence, and pointing to the pressure on the ground plane instead of wake momentum accounting for the lift (Prandtl & Tietjens 1934). However, it runs into trouble outside the light-loading approximation (as recognized by Prandtl); the flight path and the wake cannot be both parallel to the ground. This gives a choice between level flight, which makes the wake eventually approach the ground and change shape, or a level wake, which demands a descending flight so that the flow field is not truly steady.

There is also disagreement over whether the lift can be 'attributed' to the velocity induced by the trailing vortices, or by the bound vortex, or by the bound and starting vortex together. Now such attributions via the Biot-Savart law to fragments of the vortex system are somewhat unrigorous; recall that if the fragment is not a divergence-free vector field, it does not equal the curl of the velocity field produced from it by the Biot-Savart law (which does not solve $\nabla \times \boldsymbol{u} = \boldsymbol{\omega}$, but rather $\nabla^2 \boldsymbol{u} = -\nabla \times \boldsymbol{\omega}$, along with $\nabla \cdot \boldsymbol{u} = 0$). Controversies over the role of each fragment continue nevertheless, further obscured by a degree of arbitrariness in choosing the control volume, as seen below. Another unrigorous and damning feature in many treatments is the use of simple models such as planar rectangular vortex loops or sheets that are the length of the 'flight', when the far-wake is known to have rolled up, and the starting-vortex region to be very unsteady; see Moin, Leonard & Kim (1986) for the dynamics of a free hairpin vortex. The starting vortex could not be a mirror image of the bound vortex, because it is force-free. There can be no doubt that, before dissipation sets in, it is far from planar, in the (x, z)-plane as well as in the (y, z)-plane. The treatment as a planar vortex sheet (but with non-uniform descent velocity), loosely motivated by an elliptical wing loading (Sears 1974), is equally impossible to defend, and led to conclusions about pressure and velocity which are at odds with ours, and with Sears' own intuition.

The study is similar to a simpler one aimed at rotors (Spalart 2003) in that a correct and sufficiently complete description of the far field is sought, and then exploited in conservation principles. This description involves matched asymptotic expansions because the wake pierces the surfaces and therefore imposes finite velocities locally, but only after taking a simpler structure which allows detailed analysis. The inner-expansion equations are the two-dimensional Euler equations, and the outer-expansion equations the three-dimensional irrotational condition. Matched asymptotic expansions are not used as in lifting-line or lifting-surface theory (Van Dyke 1975; Moore & Saffman 1973); these have the wing region as the inner expansion, and the irrotational flow far from it, with the wake inside it, as the outer expansion. The present set and the lifting-line set could be nested, but this is not our purpose here. The state of the rolled-up wake is considered known, but the aircraft itself is not. There is no mention either of whether the aircraft is self-propelled, towed, or descending. This work overlaps with that of Batchelor (1964), but essential differences are that much of his study concerned axisymmetric vortices, or ones assumed well-separated from the mirror vortex, and that he emphasized viscous effects.

An ambition which will have to wait is a rigorous definition of induced drag in viscous flows. The concept is widely used in the industry, and it would be very valuable if a Navier–Stokes solution could be processed to rigorously separate induced drag, parasite drag, compressible drag, and finally thrust (in fact, even the division into drag and thrust is not exact). This separation is discussed below once the relevant formula is available, not that new ideas are proposed. In what follows, the wake is allowed to contain non-uniform total pressure H, which reflects propulsion and viscous drag (viscous effects are neglected only in the far wake, where gradients are weaker). Some questions are relevant only if these two effects are absent; for brevity, such an aircraft will be called a 'glider'.

The core of the paper is in § 2, with lengthy equations leading to a rather simple result, and physical discussions. After a summary in § 3, the Appendix recalls the mathematical problem of positivity of induced-drag predictions, which remains to be solved.

2. Equations

2.1. *Overview*

The coordinates are oriented so that the wake is parallel to the x-axis; the velocity vector of the aircraft with respect to the air mass is

$$-\boldsymbol{U} = -(U, 0, W) = -\sqrt{U^2 + W^2}(\cos\epsilon, 0, \sin\epsilon).$$

The angle ϵ is not simply related to downwash angles at the wing, or the direction of the force vector; it is not assumed to be small. Partial derivatives with respect to time are in the air-mass frame, and $\boldsymbol{u} = (u, v, w)$ is the local velocity vector with respect to the air mass. The y-axis is lateral and the wake symmetric in y, and the z-axis is pointing approximately upwards; observe that gravity is irrelevant. Define $r \equiv \sqrt{x^2 + y^2 + z^2}$ and $r_{2D} \equiv \sqrt{y^2 + z^2}$; the aircraft is near $r = 0$. These axes simplify the algebra, compared with flight-path axes, and the force vector will be resolved in due course to obtain lift and drag. Further, the wake is studied when fully rolled up, that is, asymptotically independent of x. The flow is also steady in the reference frame attached to the aircraft. Therefore, the operators ∂_x and $\partial_t + U\partial_x + W\partial_z$ both give zero in the far wake, implying that $\partial_t + W\partial_z$ also does: the wake is steadily descending at a velocity W.

The dominant quantities of the wake are its circulation Γ over a half-plane $y > 0$ and its effective span b_0; the centroids of vorticity ω_x in each half-plane (defined for the right half by $\iint_{y>0} y\omega_x \mathrm{d}y\,\mathrm{d}z / \iint_{y>0} \omega_x \mathrm{d}y\,\mathrm{d}z$) are at $y = \pm b_0/2$, $z = 0$: in other words $\Gamma \equiv \iint_{y>0} \omega_x \mathrm{d}y\,\mathrm{d}z$ and $\Gamma b_0 \equiv \iint y\omega_x \mathrm{d}y\,\mathrm{d}z$, while $\iint_{y>0} z\omega_x \mathrm{d}y\,\mathrm{d}z = 0$. Unless otherwise noted, such double integrals will cover the entire (y, z)-plane, and subscripts such as $y > 0$ here or $x = X$ below qualify the region of integration; b_0 is commensurate with the span b of the wing, but b plays no role here. The conservation of Γ and b_0 along the wake results from the inviscid assumption; note that real wakes have very high Reynolds numbers, and in addition that rotation tends to suppress turbulence in vortices. The wake fluid, meaning all the streamlines which carry vorticity or total-pressure differences, is assumed to be grouped within a region in the (y, z)-plane with a size of the order of b_0, called the ‘wake region’. Extreme cases in which two or more vortex pairs drift apart are excluded.

The existence of a fully rolled-up state is assumed, based on observations of smoothly loaded wings (Spalart 1996). Their wakes progress to the eventual state via stretching of the material lines in the (y, z)-plane, a process which yields axisymmetric states for single vortices. The case in which the mature state of the wake has two or more helical vortices on each side is not as rare as that of ‘parting’ pairs (Spalart 1998). It is excluded here by the assumption that $\partial_x = 0$; however, a generalization should not be very difficult by averaging over one streamwise period of the helices.

Matched asymptotic expansions are used, with the integrals calculated from a composite expansion $\boldsymbol{u}_{inner} + \boldsymbol{u}_{outer} - \boldsymbol{u}_{int}$, the subscript *int* standing for ‘intermediate’ (Van Dyke 1975). The length scales are the following. The inner scale is b_0, and in this expansion x-derivatives are neglected. The outer scale $X \gg b_0$ gives the distance from the aircraft to the plane which ends the control volume downstream of the wing. The control volume starts with an upstream plane, for instance at $x = -X$.

The starting vortex is at infinity in x. Another option would be to include a simple model of the starting vortex, at some location $X' \gg X$, in the expansion. This would have the advantage of making the two integrals of w at $x = \pm X$ each absolutely convergent, whereas here only the integral of the difference is so (§ 2.10). As a result, the control volume could be extended to $-\infty$ in x. However, including a starting vortex at finite X' makes the flow unsteady; it introduces an acceleration of order $U\Gamma b_0/X'^3$, which needs to be examined in the control volume. It turns out that the integral of the acceleration is zero, but the integral of the momentum is infinite, so that taking its time derivative would not be well-defined.

A general remark on the starting-vortex system is that it is highly unsteady, and must not be idealized as part of a rigid rectangle. This is what makes conservation of impulse unusable for drag (even for lift, it is correct only to leading order), as far as the author has found, unless it is assumed that even the region of the starting vortex has failed to roll up (as in Milne-Thomson 1958), which is incorrect. In the literature this issue forced a recourse to energy arguments, and a dichotomy (Jones 1990). This starting-vortex system is known only inasmuch as it is in a region $X' \gg X$ and the vortices which connect to it have circulation Γ and centroid spacing b_0. It is very plausible that velocities in this region are of order Γ/b_0, so that the evolution since the beginning of the flight has displaced vortex lines by distances of order $(\Gamma/Ub_0)X'$, thus creating an unknown impulse of order $\Gamma^3/b_0^2 \times X'^2/U^2$. The integral of a velocity induced by this impulse over a closed control surface of size X is then at most of order $X^3\Gamma^2/U^2b_0^2X'^2$. A consequence of this argument is that the velocity level prevailing in the starting system must be smaller than U for the structure of our model to apply at all; that is, for the flight to generate an increasing length of identical wake. The light-loading limit $\Gamma \ll Ub_0$ is a sufficient condition for this, but finite values of this ratio also allow it.

Similarly the details of the bound-vortex system, which could include a wing, a tail, rotors, and other surfaces, and is viewed as including the three-dimensional near-aircraft downwash and roll-up, are not known. The only information on this region is contained in the wake it generates at X, and the assumption that the bound-vortex system deviates from a rectangular line-vortex system with its corners at $(0, \pm b_0/2, 0)$ only by distances of order b_0. This simplified line-vortex system will be used to build the outer expansion.

2.2. *Basic integrals*

Conservation of momentum indicates that the force vector on the aircraft is

$$\boldsymbol{F} = -\iint [p\,\boldsymbol{n} + \rho((\boldsymbol{U}+\boldsymbol{u})\cdot\boldsymbol{n})(\boldsymbol{U}+\boldsymbol{u})]\,\mathrm{d}S \tag{2.1}$$

where the unit normal vector $\boldsymbol{n}$ is pointing out of the control volume, which surrounds the aircraft. Bernoulli's equation is

$$p = p_\infty + \Delta H + \rho\frac{||\boldsymbol{U}||^2 - ||\boldsymbol{U}+\boldsymbol{u}||^2}{2}. \tag{2.2}$$

The total pressure of fluid which encountered the aircraft was altered by an amount ΔH, due to parasite drag or propulsion, and this only in and near the wake, in a tubular region with a diameter of order b_0 (Batchelor 1964). Equation (2.1) becomes

$$\boldsymbol{F} = \iint \left[\left(\rho\frac{2\boldsymbol{U}\cdot\boldsymbol{u} + ||\boldsymbol{u}||^2}{2} - \Delta H\right)\boldsymbol{n} - \rho((\boldsymbol{U}+\boldsymbol{u})\cdot\boldsymbol{n})(\boldsymbol{U}+\boldsymbol{u})\right]\mathrm{d}S, \tag{2.3}$$

and conservation of mass $\iint \boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}S = 0$ brings a marginal simplification:

$$\boldsymbol{F} = \iint \left[\left(\rho\frac{2\boldsymbol{U}\cdot\boldsymbol{u} + ||\boldsymbol{u}||^2}{2} - \Delta H\right)\boldsymbol{n} - \rho((\boldsymbol{U}+\boldsymbol{u})\cdot\boldsymbol{n})\boldsymbol{u}\right]\mathrm{d}S. \tag{2.4}$$

2.3. *Control volume*

The control volume is the unbounded region between two transverse planes, which is appropriate for the following reason. A finite control volume can be chosen as a right cylinder of radius R bordered by these two planes. All the terms in the expansion will be seen to decay as fast as $1/r_{2D}^2$. As a result, the contribution to the integrals from the cylindrical part of the surface, which has an area of order R, vanishes as R increases to infinity. Therefore, it does not need to be calculated in any detail, and the task is reduced to evaluating the limit of the contributions of the two disks as R goes to infinity. The fact that these integrals are well-behaved will confirm the reasoning on the cylindrical surface. All the integrals will be versus $\mathrm{d}S = \mathrm{d}y\,\mathrm{d}z$.

2.4. *Intermediate expansion*

This expansion is introduced first, is designed in some respects for convenience, and later justified step by step. It is the velocity field $\boldsymbol{u}_{int}(x, y, z)$ induced by a dipole of strength Γb_0 placed on the x-axis, but only for $x > 0$:

$$u = 0, \tag{2.5}$$

$$v = -He(x)\,\frac{\Gamma b_0}{2\pi}\frac{2yz}{(y^2+z^2)^2}, \tag{2.6}$$

$$w = He(x)\,\frac{\Gamma b_0}{2\pi}\frac{y^2-z^2}{(y^2+z^2)^2}. \tag{2.7}$$

Subscripts such as int in u_{int} are omitted when the context allows it. For large r_{2D}, v and w are of order $\Gamma b_0/r_{2D}^2$. They contain arbitrary and in fact singular behaviour, both across $x = 0$ because of the Heaviside function He (a notation which allows the use of a global additive composite expansion) and near the centreline of the wake, for r_{2D} of order b_0. For the latter, two line vortices could have been preserved instead of a dipole, giving weaker singularities, or even regularized. However, this behaviour near the axis merely needs to match the one given the outer expansion $\boldsymbol{u}_{outer}$, so that the two cancel in the composite expansion.

2.5. *Inner expansion*

This is confined to $x > 0$, as is the intermediate expansion, and involves the quantities (u, v, w, p) as functions of y and z, with a stream function ψ such that $v = \partial\psi/\partial z$ and $W + w = -\partial\psi/\partial y$. Also, $\psi' \equiv \psi + Wy$ is the stream function of the perturbation velocities (v, w), and $\nabla^2\psi' = \nabla^2\psi = -\omega_x$. The stream function ψ is 0 on the centreline, $y = 0$, and on streamlines on each side which surround the wake and connect with the centreline. The region inside these limiting streamlines is the wake region, in which fluid is circulating; outside it, fluid is streaming by, irrotational and with unaltered total pressure. Each value of ψ identifies a streamtube, on which several quantities will be conserved: $\Delta H(\psi)$, $u(\psi)$, and $\omega_x(\psi)$ (since ψ is not necessarily monotonic, this notation is 'shorthand' for the statement that the gradients of these functions are multiples of the gradient of ψ, which is what is used mathematically). The streamwise volume flux through the wake region in this expansion is the well-defined integral $S \equiv \iint u\,\mathrm{d}y\,\mathrm{d}z$. In the light-loading approximation, $\Gamma/Ub_0 \ll 1$, this quantity is of second order, $S/Ub_0^2 = O([\Gamma/Ub_0]^2)$, but this approximation is not used.

The governing equations

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.8}$$

$$v\frac{\partial u}{\partial y} + (W+w)\frac{\partial u}{\partial z} = 0, \tag{2.9}$$

$$v\frac{\partial v}{\partial y} + (W+w)\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2.10}$$

$$v\frac{\partial w}{\partial y} + (W+w)\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{2.11}$$

are manipulated to prepare identities for later use. Eliminating p between (2.10) and (2.11) and using (2.8) gives conservation of streamwise vorticity as expected, $\omega_x = \omega_x(\psi)$, while (2.9) dictates $u = u(\psi)$. Combined with the definition of total pressure (2.2), (2.10) and (2.11) give

$$\frac{\partial}{\partial y}\left(\frac{(U+u)^2}{2} - \frac{\Delta H}{\rho}\right) = -(W+w)\omega_x = \omega_x \frac{\partial \psi}{\partial y}, \tag{2.12}$$

$$\frac{\partial}{\partial z}\left(\frac{(U+u)^2}{2} - \frac{\Delta H}{\rho}\right) = v\omega_x = \omega_x \frac{\partial \psi}{\partial z}, \tag{2.13}$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\psi}\left(\frac{(U+u)^2}{2} - \frac{\Delta H}{\rho}\right) = \omega_x, \tag{2.14}$$

showing that ΔH is also a function of ψ. The function $f \equiv Uu + u^2/2 - \Delta H/\rho$ equals $(p_\infty - p)/\rho + (W^2 - v^2 - (W+w)^2)/2$, in other words it is the Bernoulli function in the transverse plane. Further, $u = 0$ outside the wake region because of (2.14) (ω_x and ΔH being 0).

The regularity of the functions involved must be known, in view of the inviscid approximation, as pointed out by a reviewer. Vortex sheets are admitted, which is unavoidable in the glider situation (although for typical initial vortex sheets, the endless stretching mentioned earlier weakens the velocity jump and correspondingly the distance between layers of the rolled-up sheet, leading the flow field towards a continuous solution). The velocity components u, v and w may be discontinuous across these sheets; ψ is continuous, but may have jumps in its derivatives. As a result, an expression such as $v\omega_x$ in (2.13) is formally ambiguous: it could be the product of a Dirac δ function and a discontinuous function. It is in fact not ambiguous, because the two factors are related, which can be seen when (2.12) and (2.13) are re-written as

$$\frac{\partial}{\partial y}\left(\frac{(U+u)^2}{2} - \frac{\Delta H}{\rho}\right) = \frac{1}{2}\frac{\partial}{\partial y}(v^2 - w^2) + \frac{\partial}{\partial z}(vw) - W\omega_x, \tag{2.15}$$

$$\frac{\partial}{\partial z}\left(\frac{(U+u)^2}{2} - \frac{\Delta H}{\rho}\right) = \frac{1}{2}\frac{\partial}{\partial z}(w^2 - v^2) + \frac{\partial}{\partial y}(vw). \tag{2.16}$$

These expressions involve the first derivatives of discontinuous functions, but no ambiguous products. The internal structure of the vortex sheets is unspecified, but has no impact on the quantities needed, which are simple integrals of u and u^2.

Now, (2.15) and (2.16) are used to express a key integral of the velocity field. With $\boldsymbol{r}$ here the (y, z) vector, we proceed:

$$\begin{aligned}\iint f\,\mathrm{d}y\,\mathrm{d}z &= \iint f\left(\frac{\nabla\cdot\boldsymbol{r}}{2}\right)\mathrm{d}y\,\mathrm{d}z = \frac{1}{2}\int f\boldsymbol{r}\cdot\boldsymbol{n}\,\mathrm{d}s - \frac{1}{2}\iint \boldsymbol{r}\cdot\nabla f\,\mathrm{d}y\,\mathrm{d}z \\ &= -\frac{1}{2}\iint \left(y\left[\frac{1}{2}\frac{\partial}{\partial y}(v^2 - w^2) + \frac{\partial}{\partial z}(vw) - W\omega_x\right]\right. \\ &\qquad \left. + z\left[\frac{1}{2}\frac{\partial}{\partial z}(w^2 - v^2) + \frac{\partial}{\partial y}(vw)\right]\right)\mathrm{d}y\,\mathrm{d}z \end{aligned} \tag{2.17}$$

in which the line integral of $f\boldsymbol{r}\cdot\boldsymbol{n}\,\mathrm{d}s$ is written for completeness, and then dropped because outside the wake region, $f = 0$.

Most of the integrals in (2.17) are shown to be zero, as follows. The total circulation of the wake is zero by symmetry, so that v and w are of order $1/R^2$ for large R (with the structure of rational polynomials in R); the integrands are therefore of order $1/R^4$, making the integral absolutely convergent. Fubini's theorem applies, and line integrals can then be calculated in any convenient direction. This includes $\int \partial(vw)/\partial y \, dy = 0$ and $\int y \partial(v^2 - w^2)/\partial y \, dy = -\int (v^2 - w^2) \, dy$. The integrals of $(v^2 - w^2)$ and $(w^2 - v^2)$ cancel, leaving only the integral of $yW\omega_x/2$, which equals $W\Gamma b_0/2$ directly from the definition of b_0. A valuable consequence of this string of identities is that

$$\iint \left(2Uu + u^2 - \frac{2\Delta H}{\rho}\right) dy \, dz = W\Gamma b_0, \tag{2.18}$$

or alternatively

$$\iint \left(u^2 - \frac{2\Delta H}{\rho}\right) dy \, dz = W\Gamma b_0 - 2US, \tag{2.19}$$

which will be used to determine the role of u^2 in the final result.

The integral we wish to bound from above, in order to rule out the prediction of negative induced drag, is $\iint u^2 dy \, dz$; because of (2.19), this amounts to bounding S from below. Introducing the function $g \equiv \int \omega_x d\psi + \Delta H/\rho$, we have $Uu + u^2/2 = g$ and therefore $u = \sqrt{U^2 + 2g} - U$. Taking this branch of the quadratic implies $U + u > 0$, which is logical within the model of the flow; there is no reversal in the far wake. A consequence to be used below is that u has the sign of g. We also have $\partial(u^2)/\partial U = -2u^2/\sqrt{U^2 + 2g} \leqslant 0$, and it can be shown that $u^2 \leqslant 2|g|$. Unfortunately, in general, the integral of $2|g|$ is not easily related to simple quantities, nor bounded from above.

We now bound the other integral which enters the classical drag formula from below,

$$\iint (v^2 + w^2) dy \, dz = \iint \omega_x \psi' dy \, dz = \iint \omega_x \psi \, dy \, dz + W\Gamma b_0, \tag{2.20}$$

through standard identities. Note that ψ and ψ' are continuous, so that their products with ω_x are well-defined even if ω_x is singular. Now the integral of $\omega_x \psi$ can be restricted to the wake region (WR), since vorticity is zero outside it. On its boundary, $\psi = 0$, leading via the standard identity $\nabla \cdot (\psi \nabla \psi) = \psi \nabla^2 \psi + \nabla \psi \cdot \nabla \psi$ to

$$\iint (v^2 + w^2) dy \, dz = \iint_{WR} ||\nabla \psi||^2 \, dy \, dz + W\Gamma b_0 \geqslant W\Gamma b_0. \tag{2.21}$$

Observe the presence of $W\Gamma b_0$ both in (2.19) and in (2.21).

An identity of Spreiter & Sacks (1951) may provide some perspective, if only by establishing that Γ and W have the same sign:

$$W\Gamma = -\iint w\omega_x dy \, dz = -\iint \left[\frac{1}{2}\frac{\partial}{\partial y}(w^2 - v^2) - \frac{\partial}{\partial z}(vw)\right] dy \, dz = \frac{1}{2}\int_{CL} w^2 dz; \tag{2.22}$$

the last integral is taken along the centreline (CL) of the wake region, the line $y = 0$.

Another property will be useful, and concerns the integral of $u\boldsymbol{u}$. Since u is a function of ψ, we can define a function h with $dh/d\psi = u$. Then, $uv = dh/d\psi \times \partial\psi/\partial z = \partial h/\partial z$ and $u(W + w) = -dh/d\psi \times \partial\psi/\partial y = -\partial h/\partial y$. Since u reaches 0 immediately outside the wake region and therefore h reaches its outside value similarly, the integral of its gradient is 0, showing that $\iint uv dy \, dz = \iint u(W + w) dy \, dz = 0$, and therefore

$$\iint u \, \boldsymbol{u} dy \, dz = \iint u^2 \, dy \, dz \, \boldsymbol{e}_x - WS\boldsymbol{e}_z, \tag{2.23}$$

where $\boldsymbol{e}_x$, $\boldsymbol{e}_z$ and $\boldsymbol{e}_z$ are the unit vectors.

Finally, the Biot-Savart law is used to relate the inner and intermediate expansions. The v component at a point (Y, Z) (with $R^2 = Y^2 + Z^2$) is given by

$$\begin{aligned} v &= -\frac{1}{2\pi}\iint \omega_x \frac{Z-z}{(Y-y)^2+(Z-z)^2}\mathrm{d}y\,\mathrm{d}z \\ &= -\frac{1}{2\pi}\iint \omega_x \left[\frac{Z}{R^2} + \frac{2ZYy+(Z^2-Y^2)z}{R^4} + O\left(\frac{b_0^2}{R^3}\right)\right]\mathrm{d}y\,\mathrm{d}z \\ &= -\frac{1}{2\pi}\frac{2ZY}{(Y^2+Z^2)^2}\iint y\omega_x\,\mathrm{d}y\,\mathrm{d}z + O\left(\frac{\Gamma b_0^2}{R^3}\right) = -\frac{\Gamma b_0}{2\pi}\frac{2ZY}{(Y^2+Z^2)^2} + O\left(\frac{\Gamma b_0^2}{R^3}\right) \end{aligned} \tag{2.24}$$

using symmetry $(\omega_x(-y, z) = -\omega_x(y, z))$ for the two integrals not containing y and the fact that the vorticity is distributed within a region of order b_0. It is seen in (2.24) and the equivalent result for w that the difference between the inner expansion and the intermediate one (2.5)–(2.7) for v and w is of order $\Gamma b_0^2/r_{2D}^3$ for large r_{2D} (each one is of order $\Gamma b_0/r_{2D}^2$). For u, the difference is zero outside the wake. This is what makes (2.5)–(2.7) the correct intermediate expansion, pending a similar property relative to the outer expansion. This difference is well-behaved for large r_{2D}, and any integrals of it are absolutely convergent thanks to the $1/r_{2D}^3$ decay. In addition, because of the continuity equation $\partial v/\partial y + \partial w/\partial z = 0$, the integral of this difference for v and for w equals 0 (for instance the line integral $\int w \mathrm{d}y$ is independent of z because of continuity, and must be zero based on the behaviour for large r_{2D}). In summary,

$$\iint (\boldsymbol{u}_{inner} - \boldsymbol{u}_{int})\,\mathrm{d}y\,\mathrm{d}z = S\boldsymbol{e}_x. \tag{2.25}$$

2.6. *Outer expansion*

The leading vortex term in the far field is the flow induced by a rectangular vortex 'hairpin' defined by Γ and b_0, starting at $(0, \pm b_0/2, 0)$ and ending at infinity. A sink term will be introduced shortly. Here, only terms with non-zero contributions to far-field integrals are considered.

The hairpin vortex is, of course, not an accurate description of the vortex system near the aircraft, which is not located precisely at $(x, z) = (0, 0)$. The initial roll-up and steeper descent under the influence of the bound vorticity are unspecified, but the difference between the true vortex system and the hairpin vortex is finite in extent. It is bounded by the circulation and length scale of the wake, so that the velocity it induces is of order $\Gamma b_0^2/R^3$ for large R. Therefore, its contribution to integrals in a plane at a distance X is of order $\Gamma b_0^2/X$, and vanishes for large X. It will be confirmed that the contributions from the hairpin vortex are invariant under translations, either in x or z; this is consistent with the fact that the location of the aircraft is not specified to within a multiple of b_0.

Using the Biot-Savart cosine law for a filament, the bound vortex has (Milne-Thomson 1958)

$$u = \frac{\Gamma}{4\pi}\left[\frac{y+b_0/2}{\sqrt{x^2+(y+b_0/2)^2+z^2}} - \frac{y-b_0/2}{\sqrt{x^2+(y-b_0/2)^2+z^2}}\right]\frac{z}{x^2+z^2} \tag{2.26}$$

with a similar formula for w but $v=0$; the leading terms, denoted in later use by $\boldsymbol{u}_{bound}$, are

$$u = \frac{\Gamma b_0}{4\pi}\frac{z}{r^3}, \qquad v = 0, \qquad w = -\frac{\Gamma b_0}{4\pi}\frac{x}{r^3}. \tag{2.27}$$

A semi-infinite trailing line vortex at $y=b_0/2$ has

$$w = \frac{\Gamma}{4\pi}\left[1 + \frac{x}{\sqrt{x^2+(y-b_0/2)^2+z^2}}\right]\frac{y-b_0/2}{(y-b_0/2)^2+z^2} \tag{2.28}$$

with a similar formula for v, and with $u=0$.

For the vortex pair at $\pm b_0/2$, the leading terms $\boldsymbol{u}_{trail}$ for large r_{2D}/b_0 are $u=0$,

$$v = -\frac{\Gamma b_0}{4\pi}\left[\frac{x}{r^3} + \left(1+\frac{x}{r}\right)\frac{2}{y^2+z^2}\right]\frac{yz}{y^2+z^2}, \tag{2.29}$$

$$w = \frac{\Gamma b_0}{4\pi}\left[\frac{y^2}{r^2} + \left(1+\frac{x}{r}\right)\frac{y^2-z^2}{y^2+z^2}\right]\frac{1}{y^2+z^2}. \tag{2.30}$$

When in addition $r_{2D} \ll r$, that is, at small angles to the wake direction, we have $x \approx r$ and the leading terms are identical to the intermediate expansion (2.5)–(2.7). The difference between the outer and the intermediate expansion is of order $\Gamma b_0/r^2$ for large r. This is the second step in confirming the design of the intermediate expansion.

The flow $\boldsymbol{u}_{sink}$ induced by a sink at the origin is

$$u = -S'\frac{x}{4\pi r^3}, \qquad v = -S'\frac{y}{4\pi r^3}, \qquad w = -S'\frac{z}{4\pi r^3}, \tag{2.31}$$

in which S' is unspecified for now. This is again of order $1/r^2$ for large r, leading to finite integrals in the transverse planes. The associated kinetic energy is also finite. The volume flow through a plane is $\pm S'/2$, depending on the sign of x. A sink does not appear to have been involved before in infinite space, but is implicit in wind-tunnel analyses (Maskell 1972).

2.7. *Composite expansion*

This is the field $\boldsymbol{u}_{inner} + [\boldsymbol{u}_{trail} + \boldsymbol{u}_{bound} + \boldsymbol{u}_{sink}] - \boldsymbol{u}_{int}$. The orders of magnitude for large r or r_{2D} are as follows: $\boldsymbol{u}_{inner} - \boldsymbol{u}_{int}$ is confined to a region of order b_0 in the (y, z)-plane, and key integrals of it are known from § 2.5; $\boldsymbol{u}_{bound}$ and $\boldsymbol{u}_{sink}$ are of order $\Gamma b_0/r^2$; $\boldsymbol{u}_{trail}$ is of order Γ/b_0 near the wake and of order $\Gamma b_0/r_{2D}^2$ for large r_{2D}. However, it can be verified that $\boldsymbol{u}_{trail} - \boldsymbol{u}_{int}$ is of order $\Gamma b_0/r^2$.

These known magnitudes lead to cancellations. The integral of the product of a term that is confined in the (y, z)-plane and a term that is of order $1/r^2$, and therefore $1/X^2$, vanishes for large X. This in contrast to the integral of the sink term (2.31), for instance, which has a peak of order S'/X^2 but a dependence on (y, z) that is not confined; it expands for larger X. This provides a rigorous reason for dropping many of the products, which has often been done on an intuitive basis instead.

Mass conservation will be imposed, with the deviation from free-stream mass flow split into two parts $\dot{m}_{UP}$ and $\dot{m}_{DP}$, for the upstream and downstream planes respectively. The integral (2.4) is similarly split into $\boldsymbol{F}_{UP}$, which contains the contributions at the upstream plane, and $\boldsymbol{F}_{DP}$, which contains the contributions at the downstream plane. In the subsections devoted to each of the planes, the linear (as opposed to products) contribution of the trailing vortices is left unfinished, as it needs to be taken together for the two planes, in § 2.10.

2.8. *Upstream plane, $x = -X$*

In this plane, $\boldsymbol{n} = -\boldsymbol{e}_x$, and the only terms are $\boldsymbol{u}_{bound}$, $\boldsymbol{u}_{trail}$, and $\boldsymbol{u}_{sink}$. Because they are of order $1/r^2$, only linear terms in (2.4) survive as $X \to \infty$, and

$$\dot{m}_{UP} = -\rho \iint_{x=-X} (\boldsymbol{u}_{bound} + \boldsymbol{u}_{trail} + \boldsymbol{u}_{sink}) \cdot \boldsymbol{e}_x \, \mathrm{d}y \, \mathrm{d}z = -\rho \, \frac{S'}{2} \tag{2.32}$$

because of z-symmetry in u_{bound}, and because $u_{trail} = 0$. Also,

$$\boldsymbol{F}_{UP} = \rho \iint_{x=-X} [-(\boldsymbol{U} \cdot (\boldsymbol{u}_{bound} + \boldsymbol{u}_{trail} + \boldsymbol{u}_{sink}))\, \boldsymbol{e}_x + U(\boldsymbol{u}_{bound} + \boldsymbol{u}_{trail} + \boldsymbol{u}_{sink})] \, \mathrm{d}y \, \mathrm{d}z. \tag{2.33}$$

Two of the integrals are simple:

$$\iint \boldsymbol{u}_{bound} \, \mathrm{d}y \, \mathrm{d}z = \frac{\Gamma b_0}{2} \boldsymbol{e}_z, \quad \text{and} \quad \iint \boldsymbol{u}_{sink} \, \mathrm{d}y \, \mathrm{d}z = \frac{S^1}{2} \boldsymbol{e}_x\,;$$

these will be used again. The first one was used by Prandtl long ago (Prandtl & Tietjens 1934), but the second one appears to enter for the first time. We recognize its higher-order status within light-loading situations; however, the induced drag is of the same order. The result is that

$$\boldsymbol{F}_{UP} = \frac{\rho \Gamma b_0}{2} (U \boldsymbol{e}_z - W \boldsymbol{e}_x) + \rho \iint_{x=-X} (U \boldsymbol{u}_{trail} - \boldsymbol{U} \cdot \boldsymbol{u}_{trail} \ \boldsymbol{e}_x) \, \mathrm{d}y \, \mathrm{d}z \tag{2.34}$$

with the integral of $\boldsymbol{u}_{trail}$ to be addressed in § 2.10.

2.9. *Downstream plane, $x = X$*

This region has numerous terms, but almost all products drop out. Here, $\boldsymbol{n} = \boldsymbol{e}_x$, leading to

$$\dot{m}_{DP} = -\rho \iint_{x=-X} (\boldsymbol{u}_{inner} - \boldsymbol{u}_{int} + \boldsymbol{u}_{bound} + \boldsymbol{u}_{trail} + \boldsymbol{u}_{sink}) \cdot \boldsymbol{e}_x \, \mathrm{d}y \, \mathrm{d}z = \rho \left(S - \frac{S'}{2} \right) \tag{2.35}$$

so that using (2.32), mass conservation naturally reduces to $S' = S$. From here on, S' will be written as S. For momentum,

$$\begin{aligned} \boldsymbol{F}_{DP} = \iint_{x=X} & (\rho [2\boldsymbol{U} \cdot (\boldsymbol{u}_{inner} + \boldsymbol{u}_{bound} + \boldsymbol{u}_{trail} + \boldsymbol{u}_{sink} - \boldsymbol{u}_{int}) \\ & + ||\boldsymbol{u}_{inner} + \boldsymbol{u}_{bound} + \boldsymbol{u}_{trail} + \boldsymbol{u}_{sink} - \boldsymbol{u}_{int}||^2]/2 - \Delta H) \boldsymbol{e}_x \\ & - \rho \ (U + u_{inner} + u_{bound} + u_{trail} + u_{sink} - u_{int}) \\ & \times (\boldsymbol{u}_{inner} + \ \boldsymbol{u}_{trail} + \boldsymbol{u}_{bound} + \boldsymbol{u}_{sink} - \boldsymbol{u}_{int}) \, \mathrm{d}y \, \mathrm{d}z. \end{aligned} \tag{2.36}$$

This is re-arranged into

$$\begin{aligned} \boldsymbol{F}_{DP} = \rho & \iint_{x=X} \boldsymbol{U} \cdot (\boldsymbol{u}_{inner} - \boldsymbol{u}_{int}) \boldsymbol{e}_x \, \mathrm{d}y \, \mathrm{d}z + \rho \iint_{x=X} \boldsymbol{U} \cdot (\boldsymbol{u}_{trail} + \boldsymbol{u}_{bound} + \boldsymbol{u}_{sink}) \boldsymbol{e}_x \, \mathrm{d}y \, \mathrm{d}z \\ & + \rho \iint_{x=X} \frac{||\boldsymbol{u}_{inner}||^2}{2} \boldsymbol{e}_x \, \mathrm{d}y \, \mathrm{d}z + \rho \iint_{x=X} \boldsymbol{u}_{inner} \cdot (\boldsymbol{u}_{trail} + \boldsymbol{u}_{bound} + \boldsymbol{u}_{sink} - \boldsymbol{u}_{int}) \boldsymbol{e}_x \, \mathrm{d}y \, \mathrm{d}z \\ & + \rho \iint_{x=X} \frac{||\boldsymbol{u}_{trail} + \boldsymbol{u}_{bound} + \boldsymbol{u}_{sink} - \boldsymbol{u}_{int}||^2}{2} \boldsymbol{e}_x \, \mathrm{d}y \, \mathrm{d}z \\ & - \rho \iint_{x=X} (U + u_{inner}) \ \boldsymbol{u}_{inner} + (u_{trail} + u_{bound} + u_{sink} - u_{int}) \ \boldsymbol{u}_{inner} \, \mathrm{d}y \, \mathrm{d}z \end{aligned}$$

$$
\begin{aligned}
&-\rho \iint_{x=X} (U + u_{inner})(\boldsymbol{u}_{trail} + \boldsymbol{u}_{bound} + \boldsymbol{u}_{sink} - \boldsymbol{u}_{int})\,\mathrm{d}y\,\mathrm{d}z \\
&-\rho \iint_{x=X} (u_{trail} + u_{bound} + u_{sink} - u_{int})(\boldsymbol{u}_{trail} + \boldsymbol{u}_{bound} + \boldsymbol{u}_{sink} - \boldsymbol{u}_{int})\,\mathrm{d}y\,\mathrm{d}z \\
&- \iint_{x=X} \Delta H \boldsymbol{e}_x\,\mathrm{d}y\,\mathrm{d}z. \qquad (2.37)
\end{aligned}
$$

in order to bring out terms of order $1/X^2$, which are $\boldsymbol{u}_{trail} - \boldsymbol{u}_{int}$, $\boldsymbol{u}_{bound}$, and $\boldsymbol{u}_{sink}$. After products of order $1/X^4$ are dropped, the remaining terms for large X are

$$
\begin{aligned}
\boldsymbol{F}_{DP} = {}& \rho \iint_{x=X} \boldsymbol{U}\cdot(\boldsymbol{u}_{inner} - \boldsymbol{u}_{int})\boldsymbol{e}_x\,\mathrm{d}y\,\mathrm{d}z + \rho \iint_{x=X} \boldsymbol{U}\cdot(\boldsymbol{u}_{trail} + \boldsymbol{u}_{bound} + \boldsymbol{u}_{sink})\boldsymbol{e}_x\,\mathrm{d}y\,\mathrm{d}z \\
&+\rho \iint_{x=X} \left(\frac{||\boldsymbol{u}_{inner}||^2}{2}\boldsymbol{e}_x - (U + u_{inner})\boldsymbol{u}_{inner} - U(\boldsymbol{u}_{trail} + \boldsymbol{u}_{bound} + \boldsymbol{u}_{sink} - \boldsymbol{u}_{int}) \right)\mathrm{d}y\,\mathrm{d}z \\
&- \iint_{x=X} \Delta H \boldsymbol{e}_x\,\mathrm{d}y\,\mathrm{d}z. \qquad (2.38)
\end{aligned}
$$

The terms are now expressed from known quantities where possible. First, recall that $\iint(\boldsymbol{u}_{inner} - \boldsymbol{u}_{int})\,\mathrm{d}y\,\mathrm{d}z = S\boldsymbol{e}_x$ by (2.25). Then,

$$
\iint \boldsymbol{u}_{bound}\,\mathrm{d}y\,\mathrm{d}z = -\frac{\Gamma b_0}{2}\boldsymbol{e}_z \quad \text{and} \quad \iint \boldsymbol{u}_{sink}\,\mathrm{d}y\,\mathrm{d}z = -\frac{S}{2}\boldsymbol{e}_x
$$

as before. The term $U\boldsymbol{u}_{int}$ is moved to combine with $U\boldsymbol{u}_{inner}$. The $x = X$ can be omitted in most places. This gives

$$
\begin{aligned}
\boldsymbol{F}_{DP} = {}& \frac{\rho\Gamma b_0}{2}(U\boldsymbol{e}_z - W\boldsymbol{e}_x) + \rho \iint \left(\frac{||\boldsymbol{u}_{inner}||^2}{2}\boldsymbol{e}_x - u_{inner}\,\boldsymbol{u}_{inner} \right)\mathrm{d}y\,\mathrm{d}z \\
&+\rho \iint_{x=X} ((\boldsymbol{U}\cdot\boldsymbol{u}_{trail})\,\boldsymbol{e}_x - U\boldsymbol{u}_{trail})\,\mathrm{d}y\,\mathrm{d}z - \iint \Delta H \boldsymbol{e}_x\,\mathrm{d}y\,\mathrm{d}z. \qquad (2.39)
\end{aligned}
$$

The integral of $u_{inner}\boldsymbol{u}_{inner}$ is known from (2.23), finally giving

$$
\begin{aligned}
\boldsymbol{F}_{DP} = {}& \frac{\rho\Gamma b_0}{2}(U\boldsymbol{e}_z - W\boldsymbol{e}_x) + \rho \iint \frac{||\boldsymbol{u}_{inner}||^2}{2}\mathrm{d}y\,\mathrm{d}z\boldsymbol{e}_x - \iint u_{inner}^2\,\mathrm{d}y\,\mathrm{d}z\;\boldsymbol{e}_x + \rho W S\boldsymbol{e}_z \\
&+\rho \iint_{x=X} (\boldsymbol{U}\cdot\boldsymbol{u}_{trail}\boldsymbol{e}_x - U\boldsymbol{u}_{trail})\,\mathrm{d}y\,\mathrm{d}z - \iint \Delta H \boldsymbol{e}_x\,\mathrm{d}y\,\mathrm{d}z. \qquad (2.40)
\end{aligned}
$$

2.10. *Joint integral of $\boldsymbol{u}_{trail}$ over the two planes*

The contributions left unresolved in (2.34) and (2.40) are

$$
\boldsymbol{F}_{trail} \equiv \rho \iint_{x=-X} (U\boldsymbol{u}_{trail} - \boldsymbol{U}\cdot\boldsymbol{u}_{trail}\;\boldsymbol{e}_x)\,\mathrm{d}y\,\mathrm{d}z + \rho \iint_{x=X} (\boldsymbol{U}\cdot\boldsymbol{u}_{trail}\boldsymbol{e}_x - U\boldsymbol{u}_{trail})\,\mathrm{d}y\,\mathrm{d}z \qquad (2.41)
$$

or

$$
\boldsymbol{F}_{trail} \equiv \rho \iint (U\Delta\boldsymbol{u}_{trail} - \boldsymbol{U}\cdot\Delta\boldsymbol{u}_{trail})\;\boldsymbol{e}_x\,\mathrm{d}y\,\mathrm{d}z \qquad (2.42)
$$

where $\Delta\boldsymbol{u}_{trail} \equiv \boldsymbol{u}_{trail}(-X, y, z) - \boldsymbol{u}_{trail}(X, y, z)$. We consider only the integral of w, since in $\Delta\boldsymbol{u}_{trail}$, $u = 0$ and v integrates to 0 by symmetry. It is given by

$$\Delta w_{trail} = \frac{\Gamma b_0}{4\pi}\left[\frac{y^2}{(X^2+y^2+z^2)} + \left(1 - \frac{X}{\sqrt{X^2+y^2+z^2}}\right)\frac{y^2-z^2}{y^2+z^2} - \frac{y^2}{(X^2+y^2+z^2)} - \left(1 + \frac{X}{\sqrt{X^2+y^2+z^2}}\right)\frac{y^2-z^2}{y^2+z^2}\right]\frac{1}{y^2+z^2} \quad (2.43)$$

or

$$\Delta w_{trail} = -\frac{\Gamma b_0 X}{2\pi}\frac{1}{\sqrt{X^2+y^2+z^2}}\frac{y^2-z^2}{(y^2+z^2)^2}. \quad (2.44)$$

This integral provides absolute convergence since the integrand is of order $1/r_{2D}^3$, and when written in polar coordinates it is proportional to $\int_0^{2\pi}(\cos^2\theta - \sin^2\theta)\,\mathrm{d}\theta$, which equals zero. Therefore, the joint integral is zero.

2.11. *Final result*

Adding $\boldsymbol{F}_{UP}$ and $\boldsymbol{F}_{DP}$ produces a simpler formula:

$$\begin{aligned}\boldsymbol{F} &= \rho\Gamma b_0(U\boldsymbol{e}_z - W\boldsymbol{e}_x) + \rho W S\boldsymbol{e}_z \\ &\quad + \rho\iint_{x=X}\frac{v_{inner}^2 + w_{inner}^2 - u_{inner}^2}{2}\,\mathrm{d}y\,\mathrm{d}z\,\boldsymbol{e}_x \\ &\quad - \iint_{x=X}\Delta H\,\mathrm{d}y\,\mathrm{d}z\,\boldsymbol{e}_x.\end{aligned} \quad (2.45)$$

Recall that the x-axis is aligned with the wake, not the flight path; the equivalent expression in flight axes is less readable, except for the first term. The first line starts with the classical lift, $-\rho\Gamma b_0(\boldsymbol{e}_y \times \boldsymbol{U})$, and ends with a higher-order (in the light-loading regime) modification which carries a small amount of lift and drag. The third contains parasite drag and thrust; in cruise flight, it will allow $\boldsymbol{F}$ to be aligned with gravity. The second line repeats the traditional integral for induced drag, but directed along the x-axis instead of the flight direction; this entails a loss of lift. This general agreement is favourable in terms of consistency, but defeats the hope of removing the negative sign for $u^2/2$. Fortunately, its presence will be reconciled with kinetic-energy considerations shortly; the issue of whether negative induced drag could be predicted will be taken up as well.

In formula (2.45), the lift can be formally traced in the integrals to the bound vortex, with half in (2.34) and half in (2.40). This means attributing half of the lift to upwash upstream of the aircraft and half to downwash downstream of it, just like in two dimensions. However, if the vortex system is terminated at a finite distance X', no matter how large, the upstream integral vanishes, and the downstream integral accounts for all the lift (L. B. Wigton, private communication), which to some is intuitively more satisfying. This dependence on a detail suggests that the relevance of the share between these two planes is low. On the other hand, the formal connection with the bound vortex and possibly the starting vortex, rather than the trailing vortices, is clear and contrasts with some established interpretations (§ 1).

2.12. *Analysis via the kinetic energy*

Guided by classical arguments, an educated observer might expect that, possibly within factors such as $\cos\epsilon$, the induced drag equals the energy deposited by the airplane per unit length of the wake; this contains $\rho(u^2+v^2+w^2)/2$. This kinetic energy is relative to the air mass; an energy statement in the frame of the aircraft can be derived, but is not very helpful.

In order to explore the kinetic energy the control volume, the half-space $x < X$, is now fixed with respect to the atmosphere. The steady flow pattern of the aircraft is travelling at a velocity U normal to the transverse plane and therefore from kinematics

$$\rho U \iint_{x=X} \frac{||\boldsymbol{u}||^2}{2}\,\mathrm{d}y\,\mathrm{d}z = \rho\,\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{x<X} \frac{||\boldsymbol{u}||^2}{2}\,\mathrm{d}V = \rho \iiint_{x<X} \boldsymbol{u}\cdot\frac{\partial \boldsymbol{u}}{\partial t}\,\mathrm{d}V \qquad (2.46)$$

with the time derivative in the third integral also in the frame of the atmosphere. We use the momentum equation $\partial\boldsymbol{u}/\partial t = \boldsymbol{u}\times\boldsymbol{\omega} - \nabla(p/\rho + ||\boldsymbol{u}||^2/2)$ to reformulate the right-hand side of (2.46). First, $\boldsymbol{u}\cdot(\boldsymbol{u}\times\boldsymbol{\omega}) = 0$. Second, with the vector identity $\boldsymbol{u}\cdot\nabla(p/\rho + ||\boldsymbol{u}||^2/2) = \nabla\cdot((p/\rho + ||\boldsymbol{u}||^2/2)\boldsymbol{u}) - (p/\rho + ||\boldsymbol{u}||^2/2)\nabla\cdot\boldsymbol{u}$ and the continuity condition $\nabla\cdot\boldsymbol{u} = 0$, we are left with the integral in the plane of $(p/\rho + ||\boldsymbol{u}||^2/2)\boldsymbol{u}\cdot\boldsymbol{n}$: (2.46) becomes

$$\rho U \iint_{x=X} \frac{||\boldsymbol{u}||^2}{2}\,\mathrm{d}y\,\mathrm{d}z = P - \iint_{x=X}\left(p - p_\infty + \rho\frac{||\boldsymbol{u}||^2}{2}\right) u\,\mathrm{d}y\,\mathrm{d}z \qquad (2.47)$$

where P is the power applied to the fluid in the region $x < X$ by the aircraft in an unspecified manner (much like the manner in which it applies a force onto the fluid). The last integral contains the total pressure, but now with respect to the atmosphere, and represents the flux of mechanical energy across the transverse plane. This flux defeats the educated observer's tentative identity. This is re-arranged, and the use of Bernoulli's equation leads to alternative forms for P:

$$\begin{aligned} P &= \iint\left(\rho U\frac{||\boldsymbol{u}||^2}{2} + \left(p - p_\infty + \rho\frac{||\boldsymbol{u}||^2}{2}\right) u\right)\mathrm{d}y\,\mathrm{d}z \\ &= \iint\left(\rho U\frac{||\boldsymbol{u}||^2}{2} + (\Delta H - \rho\boldsymbol{U}\cdot\boldsymbol{u})\,u\right)\mathrm{d}y\,\mathrm{d}z. \end{aligned} \qquad (2.48)$$

In summary the half-space $x < X$ is increasing its atmosphere-frame kinetic energy at the expected rate $\rho U/2 \iint ||\boldsymbol{u}||^2\mathrm{d}y\,\mathrm{d}z$, but it is also expelling fluid into the wake region with low total pressure (again in the reference frame of the atmosphere) $p + \rho||\boldsymbol{u}||^2/2$, while accepting from fluid all around with higher total pressure, due to the sink term. The region $x > X$, or 'starting region', is surrendering energy to the region $x < X$, or 'aircraft region'. As a result, the power supplied by the aircraft is lower than the rate of increase of the kinetic energy integrated over the region $x < X$. Energy, created earlier, enters the aircraft region.

For a system without total-pressure changes, $\Delta H = 0$, the induced drag is $D_i \equiv \boldsymbol{F}\cdot\boldsymbol{U}/||\boldsymbol{U}||$, for which (2.45) gives

$$D_i = \sin\epsilon\rho W S + \rho\cos\epsilon \iint\left(\frac{||\boldsymbol{u}_{inner}||^2}{2} - u_{inner}^2\right)\mathrm{d}y\,\mathrm{d}z \qquad (2.49)$$

so that the induced power $P_i \equiv D_i||\boldsymbol{U}||$ is

$$P_i = \rho W^2 S + \rho U \iint\left(\frac{||\boldsymbol{u}_{inner}||^2}{2} - u_{inner}^2\right)\mathrm{d}y\,\mathrm{d}z. \qquad (2.50)$$

This identity can also be obtained from (2.48), using the composite expansion and (2.23). The factor $\cos\epsilon$ in (2.49) comes from the fact that the distance travelled by the aircraft is the length of wake along the x-axis, divided by $\cos\epsilon$. This factor cancels in (2.50). In summary, the kinetic energy is accounted for, without any paradox attached to the $-u^2/2$ term.

2.13. *Positivity of induced drag*

This is restricted to 'gliders'. Recall that (2.19) and (2.21) showed that the integral of $(v^2 + w^2 - u^2)$ is larger than $2US$. Further, the drag contribution of the term $\rho W S \boldsymbol{e}_z$ in (2.45) is $\rho S \sin^2 \epsilon \sqrt{U^2 + W^2}$ and therefore has the sign of S. Thus, showing that $S \geqslant 0$, i.e. that the axial flow is globally jet-like provided the total pressure is uniform, would be sufficient to ensure that the predicted drag is positive.

This proof is possible if the vorticity is all of the same sign on each side, say $\omega_x \geqslant 0$ for $y > 0$. In such a case, $\psi \geqslant 0$ over the entire half of the wake region, because $\psi = 0$ on its boundary, and $\nabla^2 \psi \leqslant 0$, so that ψ cannot have a local minimum. Then, the function $g \equiv \int \omega_x \, \mathrm{d}\psi$ of § 2.5 must also be positive and therefore u, which has the sign of g, is positive. As a result its integral, S, is positive. Visually, this comes from vortex lines all carrying vorticity of the same sign, and taking helical shapes handed in the same direction.

The general case in which ω_x takes both signs has not been resolved at this point. This is unfortunate, as most airplanes shed opposite vorticity from their horizontal tails, and in an inviscid model of the flow this vorticity persists, interleaved with the primary vorticity. The obstacle is that $u = \sqrt{U^2 + 2g} - U$ is a convex function of g, making it difficult to bound the integral of u from below even though the integral of g is known. All that is known is that g must remain above $-U^2/2$, and its integral $W\Gamma b_0/2$ is positive. In a search for a counter-example to positivity, it is a simple matter to devise a g function and a U value which would make S negative, but another matter to devise functions ψ and ω_x related by $\nabla^2 \psi = -\omega_x$ such that $g = \int \omega_x \, \mathrm{d}\psi$ leads to negative S. Even then, these functions are likely to add distance in the inequality (2.21), defeating the attempt at a complete counter-example.

The deeper issue is that wake flow fields which can be the end result of the roll-up process behind an aircraft might obey constraints beyond those listed here, which are only the Euler and Bernoulli equations. The former are local to the wake; the latter is the only aspect of the history of the fluid which is accounted for. A more comprehensive analysis than the present one may lead to such a constraint, and the constraint ensure positive predicted drag.

An argument could be made that opposite vorticity is normally mixed into the dominant vorticity, because it triggers the centrifugal instability; therefore, the single-sign case would be sufficient. This argument fails, because such a flow would have suffered turbulent dissipation, and therefore not be representative of a glider any more.

An intriguing question is whether the inequality (2.21) is tight, or whether stronger results exist; for the classical elliptically loaded wing, (2.21) is over-satisfied by a large ratio, near 5. If it is tight, what kind of wake might produce lower induced drag than the classical theory predicts, for a meaningful set of constraints? This requires low $\iint \omega_x \psi \, \mathrm{d}y \, \mathrm{d}z$, which is achieved if all the vorticity ω_x is carried by a streamtube close to the dividing streamtube on which $\psi = 0$; i.e. the support of vorticity is a shell. Thus, the inequality does appear to be tight, although these distributions are far from realistic. They minimize the transverse kinetic energy, and maximize u: the inside of the shell becomes a large jet. This in turn boosts S (the integral of u, at the expense of that of u^2), again raising the predicted drag. This bind is illustrated by rewriting (2.49) as

$$D_i = \rho \sqrt{U^2 + W^2} (2 \cos \epsilon + \sin^2 \epsilon) \, S + \rho \cos \epsilon \iint \omega_x \psi \, \mathrm{d}y \, \mathrm{d}z. \qquad (2.51)$$

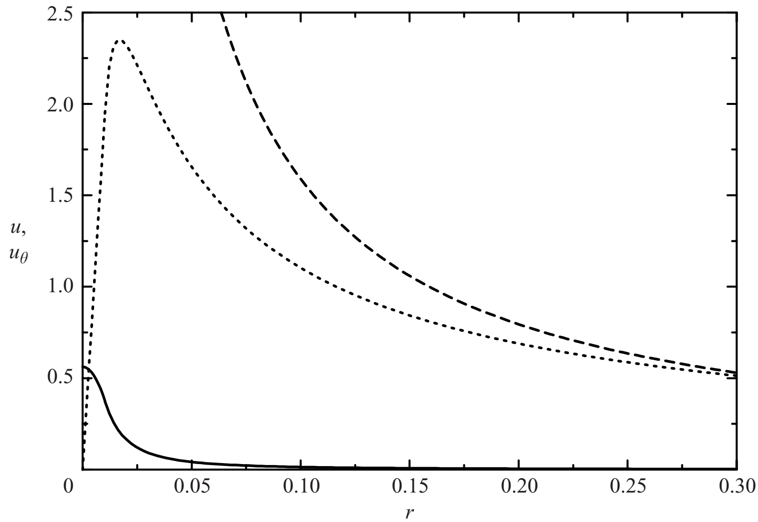

FIGURE 1. Velocity profiles in empirical model (2.52), with $\Gamma/Ub_0 = 0.05$. $\cdots$, u_θ; - - -, $\Gamma/2\pi r$; —, u; all normalized with Γ and b_0.

Recall that the second term is positive. The detailed mathematical problem which remains open is defined in the Appendix. It is proposed to minimize the ratio of D_i to $\rho\Gamma^2$, which is a valid problem, but not derived from the true design priorities for aircraft.

Also intriguing is the symmetry of vorticity distributions in z. Roll-up simulations have all led to such a symmetry, but this does not prove that asymmetric patterns cannot satisfy the equations (they might also exist, and be unstable).

2.14. *Magnitude of the u^2 effect*

An estimate is made of the magnitude of the u^2 term, using an analytical model which approaches numerical results for an elliptical spanloading (Spalart 1996):

$$u_\theta(r) = \frac{\Gamma}{2\pi r} \, \min\left(1, 1.27 + \frac{1}{4}\log\left(\frac{r}{b_0}\right)\right) \tag{2.52}$$

for $r/b_0 > 0.0103$, with solid-body rotation in the small region, $r/b_0 < 0.0103$. Here, r is the distance from the vortex axis. From u_θ, fair approximations of ψ and ω_x and therefore g and u are obtained, neglecting the interaction with the other vortex ($g \approx -(\Gamma/16\pi^2)\int(1.27 + 1/4\ \log(r/b_0))/r^3\,\mathrm{d}r$). For a typical airliner cruise condition with $\Gamma/(Ub_0) = 0.05$, u peaks at $0.56U$ which is far from negligible as seen in figure 1. However its extent is small, so that S equals only $0.0074\Gamma b_0$, and the integral of u^2 is $0.0004\Gamma^2$ so that the drag reduction normalized with the standard induced drag $\pi\Gamma^2/8$ is 0.1 %. This is undetectable. In high lift with $\Gamma/(Ub_0) = 0.2$, it is 1.4 %, or about 1 % of total drag, which is detectable. The peak u is near $2U$, but this is dependent on arbitrary details of the model. The integral is less dependent, but the estimate should still be taken with an uncertainty of the order of 25 %, especially since the model derives from two-dimensional calculations which become unrealistic with higher loadings.

3. Summary

The analysis presented here possesses attractive features. The initial hope of 'correcting' the sign of u^2 was not fulfilled, but the apparent paradox, relative to the true kinetic energy, was resolved. The final result (2.45) yields lift and drag and is almost as simple as the classical ones, which were dependent on the light-loading approximation; the new term $\rho W S \boldsymbol{e}_z$ and the re-direction of the induced-drag term are of higher order than the leading drag term, so that the old and new theories are not in conflict.

It was demonstrated that the formula cannot predict negative drag for a glider with a monotonic circulation distribution (which produces single-signed vorticity), thus not fully removing a doubt which has been lingering, at least in the author's mind. This incomplete proof of positivity applies to old and new theory. It is unfortunately rather involved; a simpler one may be found, as may a general one for vorticity of both signs.

The finding from (2.14) that the streamwise velocity perturbation u is exactly zero outside the wake (meaning the region with vorticity and/or total-pressure alteration), instead of approaching zero like a power of r in the way that v and w do, was also unexpected. It is actually easy to justify, since variations of u require non-zero ω_y or ω_z.

In (2.45) it is tempting to define the induced force (primarily, drag) as the second line, and the combination of viscous losses and propulsion as the third line. The first line would contain the lift, and a small correction. This definition would be unwise, inasmuch as the flow around 'a glider which would produce the same second line as the true aircraft in (2.45)' is a virtual entity. In particular, u_{inner} responds strongly to ΔH, thus invading the second line, which prompted Betz to introduce a virtual velocity u^* which would prevail if ΔH were absent and (v, w, p) the same in (2.2) (Maskell 1972). This is a much more cogent idea, but this wake generated by the 'Betz glider' is still somewhat virtual (it also fails to conserve mass, requiring an extensive inspection of the steps which demand that conservation). In addition, such a definition would not be resistant to residual dissipation; it is a common occurrence with tentative definitions of induced drag in slightly dissipative flows that the apparent induced drag and the apparent parasite drag swap when measurements are taken in different transverse planes. As a result, the argument that (whether experimentally or numerically) obtaining the forces from wake surveys rather that at the aircraft allows an instructive decomposition remains weak.

In the future, the various identities derived here may be used in a search for more effective definitions of induced drag from wake surveys, i.e. more tolerant of dissipation and/or incomplete roll-up. In fact any compelling general definition of induced drag, even one using the entire flow field, would be very welcome.

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Appendix

The mathematical problem expressing the positivity issue is recalled here. The solution consists of a vorticity field $\omega_x(y, z)$, antisymmetric in y and zero outside a compact domain, and a scalar U. Additional quantities are defined by:

$$\psi' = \nabla^{-2}\omega_x \text{ with } \psi' = O(1/r)$$

for large r;

$$w = -\partial\psi'/\partial y; \quad \Gamma = \iint_{y>0} \omega_x \,\mathrm{d}y\,\mathrm{d}z; \quad W = -\iint_{y>0} w\omega_x \,\mathrm{d}y\,\mathrm{d}z/\Gamma;$$
$$\epsilon = \tan^{-1}(W/U); \quad \psi = \psi' - Wy.$$

The fields satisfy $\nabla\omega_x \times \nabla\psi = 0$. Then, $g = \int \omega_x d\psi$; $u = \sqrt{U^2 + 2g} - U$. Finally, the glider drag is

$$D/\rho = (2U + W\sin\epsilon)\iint u \,\mathrm{d}y\,\mathrm{d}z + \cos\epsilon \iint \omega_x\psi \,\mathrm{d}y\,\mathrm{d}z.$$

The principal question is whether D can be negative; if not, what is the minimum value of $D/(\rho\Gamma^2)$. Identities given in § 2.5, particularly (2.19), may be used to express D differently.

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